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Strong Coapproximation in Banach Spaces

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1. INTRODUCTION

Let M be a nonempty proper subset of a normed linear space X. Without loss of generality we shall assume in the following that M is not a singleton. Following Papini and Singer [12], an element $m \in M$ is said to be a *best coapproximation in M* to an element $x \in X$ if

$$\|m-y\| \leqslant \|x-y\| \tag{1.1}$$

for all y in M. The set (perhaps empty) of all such elements m is denoted by $\mathscr{R}_M(x)$. Moreover, let $\mathfrak{D}_M$ be the set of all $x \in X$ such that $\mathscr{R}_M(x) \neq \varnothing$. Clearly, we have $\mathfrak{D}_M \supset M$. We note that this kind of "approximation" has been introduced by Franchetti and Furi [8], and that several of its properties have been established in Refs. [8, 11, 12].

Throughout this paper we shall assume that g is an increasing convex function defined on the interval $[0, \infty)$ and such that $g(0)=0$. An element $m \in M$ is said to be a *strong coapproximation* in M to an element $x \in X$ (with respect to g) if there exists a constant $c = c(x) > 0$ such that the inequality

$$g(\|m-y\|) \leqslant g(\|x-y\|) - cg(\|x-m\|) \tag{1.2}$$

holds for all y in M. Note that strong coapproximations with respect to $g(s)=s$ were first studied by Papini [11]. Denote the set of all elements x in X having the strong coapproximation m in M by $\mathfrak{D}_M^0$. Clearly, we have $\mathfrak{D}_M \supset \mathfrak{D}_M^0 \supset M$. A positive constant c_g is called an *absolute coapproximation constant* if $c(x) \geqslant c_g$ for all x in $\mathfrak{D}_M^0$.

It is evident that the strong coapproximation $m \in M$ to x is the best coapproximation in M to x. In this paper we show that the converse

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statement is also valid under some additional assumptions about M and X. Moreover, we compute absolute coapproximation constants for a number of classical Banach spaces X.

2. Cosuns in Best Coapproximation

A subset M of X is called a *cosun* if $m \in \mathscr{R}_M(x)$ implies $m \in \mathscr{R}_M(m + t(x-m))$ for every $t > 0$. By (1.1) we readily conclude that M is a cosun if and only if the inequalities

$$\|m - y\| \leqslant \|[(1-t)\,m + tx] - y\|, \qquad y \in M, \tag{2.1}$$

hold for all $t > 0$, $x \in \mathfrak{D}_M$, and $m \in \mathscr{R}_M(x)$. Note that an affine subset $M = z + N$ of X is a cosun for any linear subspace N of X and element $z \in X$. This follows immediately from the fact that inequalities (2.1) are equivalent to the inequalities

$$\|m - [(1-s)\,m + sy]\| \leqslant \|x - [(1-s)\,m + sy]\|, \qquad y \in M,$$

with $s = 1/t > 0$ which one can obtain by setting $y = (1-s)\,m + sy \in M$ into inequality (1.1). Now we show that cosuns play the same role in the theory of best coapproximation as suns in the theory of best approximation (see [2, 6, 17]). For this purpose, we define

$$\tau_g(x, y) = \tau_{g,X}(x, y) := \lim_{t \to 0+} [g(\|x + ty\|) - g(\|x\|)]/t \tag{2.2}$$

for any x, $y \in X$. In the particular case when $g(t) = t^p$, we shall write τ_p instead of τ_g. Similarly as in [4, Lemma 1, p. 446], one can show (see [17, Lemma 2.1]) that the right derivative $\tau_g(x, y)$ exists and

$$\tau_g(x, y) \leqslant [g(\|x + sy\|) - g(\|x\|)]/s \leqslant [g(\|x + ty\|) - g(\|x\|)]/t \tag{2.3}$$

for any x, $y \in X$ and $0 < s \leqslant t$.

THEOREM 2.1. *Let M be a cosun in X. Then an element $m \in M$ is a best coapproximation in M to an element $x \in X$ if and only if*

$$\tau_g(m - y, x - m) \geqslant 0 \tag{2.4}$$

for all y in M.

Proof. If $m \in \mathscr{R}_M(x)$, then it follows from (2.1) that

$$[g(\|m - y + t(x-m)\|) - g(\|m - y\|)]/t \geqslant 0 \tag{2.5}$$

for all $y \in M$ and $t > 0$. Hence by (2.2) we get (2.4). Conversely, if (2.4) holds then we can use (2.3) to get inequality (2.5) which is equivalent to (1.1) in the case when $t = 1$. ∎

It should be noticed that this theorem coincides with Proposition 1 of Papini [11] in the case when $g(s) = s$. The hypothesis in Theorem 2.1 cannot be weakened in general. Indeed, if M is not a cosun then in view of (2.1), there exist a real $t > 0$, $x \in \mathfrak{D}_M$, and $m \in \mathscr{R}_M(x)$ such that

$$[g(\|m - y + s(x - m)\|) - g(\|m - y\|)]/s < 0 \tag{2.6}$$

for $y \in M \backslash \{m\}$ and $s = t$. Hence by (2.3) we conclude that (2.6) is valid also for all $s \in (0, t)$. Therefore, letting $s \to 0+$ in (2.6), we get

$$\tau_g(m - y, x - m) < 0.$$

This in conjunction with Theorem 2.1 gives the following characterization of cosuns.

THEOREM 2.2. *A subset M of X is a cosun if and only if*

$$\inf_{y \in M} \tau_g(m - y, x - m) \geqslant 0$$

for all $x \in \mathfrak{D}_M$ and $m \in \mathscr{R}_M(x)$.

3. Strong Coapproximation in Hardy, Lebesgue, Sobolev, and Hilbert Spaces

Let X_p be the $L_p(S, \Sigma, \mu)$ space [4], Hardy space H^p [5], or Sobolev space $H^{k,p}(T)$ [1], where $1 < p < \infty$, (S, Σ, μ) is a positive measure space, $k \geqslant 0$, and T is an open subset of $\mathbb{R}^n$. In [15, 17] we have proved that there exist positive constants c_p such that inequalities

$$\tau_p(y, x - y) \leqslant \|x\|^p - \|y\|^p - c_p \|x - y\|^p, \qquad 2 \leqslant p < \infty, \tag{3.1}$$

and

$$[\tfrac{1}{2}(\|x\| + \|y\|)]^{2-p}[\|x\|^p - \|y\|^p - \tau_p(y, x - y)] \geqslant c_p \|x - y\|^2, \quad 1 < p < 2, \tag{3.2}$$

are valid for all x, y in X_p. The constants c_p satisfy the estimates

$$2^{p-2} < c_p < (p - 1)\, 2^{2-p}, \qquad p > 2, \tag{3.3}$$

and

$$2^{p-3}p(p-1) < c_p < p(p-1)/2, \qquad 1 < p < 2. \tag{3.4}$$

Moreover, we have $c_2 = 1$. Clearly, inequalities (3.1), (3.2) hold if we take c_p equal to the lower bounds given in (3.3), (3.4).

THEOREM 3.1. *Let M be a cosun in the space X_p, $1 < p < \infty$, and let m be a best coapproximation in M to an element $x \in X_p$. Then the inequality*

$$\|m-y\|^q \leqslant \|x-y\|^q - c_p \|x-m\|^q \tag{3.5}$$

holds for all y in M, where $q = \max(2, p)$.

Proof. The substitution of $x-y$ for x and $m-y$ for y into inequality (3.1) and application of Theorem 2.1 yields (3.5) in the case when $p \geqslant 2$. If $1 < p < 2$, then the same substitution into inequality (3.2) and application of Theorem 2.1 implies that

$$[\tfrac{1}{2}(\|x-y\| + \|m-y\|)]^{2-p}(\|x-y\|^p - \|m-y\|^p) \geqslant c_p \|x-m\|^2$$

for all y in M. Hence we can apply the inequality

$$t^2 - s^2 \geqslant (t^p - s^p)\left(\frac{t+s}{2}\right)^{2-p}, \qquad t \geqslant s \geqslant 0, \quad 1 < p < 2,$$

given in [17, Lemma 3.2], in order to complete the proof. ∎

Note that Theorem 3.1 shows that if M is a cosun in the space X_p, then the best coapproximation $m \in \mathscr{R}_M(x)$ is the strong coapproximation (with respect to $g(t) = t^q$) in M to each x in $\mathfrak{D}_M = \mathfrak{D}_M^0$, and c_p is an absolute coapproximation constant. This theorem can be extended to the class of uniformly convex Banach spaces which have the modulus of convexity of power type $q \geqslant 2$ [9, p. 63]. We recall that a uniformly convex space X has the modulus of convexity δ_X of power type $q \geqslant 2$ if there exists a constant $d > 0$ such that

$$\delta_X(\varepsilon) \geqslant d\varepsilon^q, \qquad 0 < \varepsilon \leqslant 2.$$

THEOREM 3.2. *Let M be a cosun in a uniformly convex space X with the modulus of convexity of power type $q \geqslant 2$, and let m be a best coapproximation in M to an element $x \in X$. Then*

$$\|m-y\|^q \leqslant \|x-y\|^q - c\|x-m\|^q \tag{3.6}$$

for all y in M, where c is a positive constant dependent only on d and q.

Proof. By Corollary 2.2 and Lemma 2.1 presented in [13], we have

$$\tau_q(y, x-y) \leqslant \|x\|^q - \|y\|^q - c\,\|x-y\|^q \tag{3.7}$$

for all x, y in X, where c is a positive constant such that

$$\delta_{L_q(X)}(\varepsilon) \geqslant c\varepsilon^q, \qquad 0 < \varepsilon \leqslant 2. \tag{3.8}$$

Finally, replacing x by $x-y$ and y by $m-y$ in inequality (3.7), and applying Theorem 2.1 to the right-hand side, we obtain (3.6). ∎

It should be noticed that Theorem 3.2 can be applied comparatively easily to prove Theorem 3.1 (cf. [13, 16]). Unfortunately, if $X = X_p$ then the best constant c in (3.8) is equal to

$$c = \begin{cases} 2^{-p}/p, & \text{if } p \geqslant 2, \\ (p-1)/8, & \text{if } 1 < p \leqslant 2, \end{cases} \tag{3.9}$$

and so it is much smaller than the constant c_p. Moreover, by Proposition 24 of Figiel [7] it follows that Theorem 3.2 can be applied to a Banach space X, which is p-convex and s-concave with $1 < p \leqslant s < \infty$ [7]. In this case we have $q = \max(2, s)$ and

$$c = q^{-1}(\max(2, 2/(p-1)^{1/2}))^{-q}.$$

We remark that if X is an inner product space (e.g., $X = X_2$), then inequality (3.1) becomes the equality

$$\tau_2(y, x-y) = 2 \operatorname{Re}(y, x-y) = \|x\|^2 - \|y\|^2 - \|x-y\|^2, \qquad x, y \in X, \tag{3.10}$$

which can be verified directly. By inserting $x := x-y$ and $y := m-y$ into (3.10) and applying Theorem 2.1 we get

THEOREM 3.3. *Let M be a cosun in an inner product space X, and let m be a best coapproximation in M to an element $x \in X$. Then*

$$\|m-y\|^2 \leqslant \|x-y\|^2 - \|x-m\|^2 \tag{3.11}$$

for all y in M.

The inequality (3.11) can be rewritten in the form

$$\|x-m\|^2 \leqslant \|x-y\|^2 - \|m-y\|^2$$

for all y in M. This inequality means that if M is a cosun in an inner product space X then a best coapproximation m in M to an element $x \in X$ is a strongly unique best approximation in M to x in the sense of the definition introduced in [14]. Conversely, if M is a sun [6] in an inner product space X then a best approximation m in M to an element $x \in X$ is a strong coapproximation in M to x. This statement follows immediately from the following theorem which slightly generalizes Theorems 2.1 in [14] and 3.1 in [15].

THEOREM 3.4. *Let m be a best approximation in a sun $M \subset X$ to an element x of an inner product space X. Then*

$$\|x-m\|^2 \leqslant \|x-y\|^2 - \|m-y\|^2$$

for all y in M.

Proof. Replace x by $x-y$ and y by $x-m$ in inequality (3.10) and use the Kolmogorov criterion

$$\tau_2(x-m, m-y) \geqslant 0, \qquad y \in M,$$

for a best approximation m in a sun M (see [2, 17]). ∎

4. MIDPOINT COSUNS AND COSUNS

All results presented in previous sections remain valid if we suppose that the implication occurring in the definition of cosuns from Section 2 is true only for $t=\frac{1}{2}$. In order to show this, we introduce an auxiliary definition. A subset M of X is called a *midpoint cosun* if $m \in \mathscr{R}_M(x)$ implies $m \in \mathscr{R}_M((m+x)/2)$. By (1.1) it follows that M is a midpoint cosun if and only if the inequalities

$$\|m-y\| \leqslant \left\| \frac{m+x}{2} - y \right\|, \qquad y \in M, \tag{4.1}$$

hold for all $x \in \mathfrak{D}_M$ and $m \in \mathscr{R}_M(x)$. Clearly, a cosun is a midpoint cosun. Conversely, if M is a midpoint cosun and $m \in \mathscr{R}_M(x)$ then inequality (4.1) holds for the elements x equal to $x_1 := (m+x)/2, ..., x_k := (m+x_{k-1})/2 = (1-2^{-k})\, m + 2^{-k}x$. Hence we have

$$\begin{aligned}\|m-y\| &\leqslant (t2^k - |t2^k-1|) \left\| \frac{m+x_{k-1}}{2} - y \right\| \leqslant t2^k \|x_k - y\| \\ &\quad - |t2^k-1|\, \|m-y\| \leqslant \|t2^k(x_k-y) - (t2^k-1)(m-y)\| \\ &= \|[m+t(x-m)] - y\|\end{aligned}$$

for all $y \in M$ and $t \geqslant 2^{-k}$. By (1.1) it follows that $m \in \mathscr{R}_M(m+t(x-m))$ for every $t \geqslant 2^{-k}$, where $k = 1, 2, \ldots$. Thus M is a cosun and so the notions of cosuns and midpoint cosuns coincide.

Inequality (4.1) suggests the following new way for proving that best coapproximations are strong coapproximations.

THEOREM 4.1. *Suppose that there exists a positive constant c such that the inequality*

$$2g\left(\left\|\frac{u+v}{2}\right\|\right) \leqslant g(\|u\|) + g(\|v\|) - cg(\|u-v\|) \tag{4.2}$$

holds for all u, v in X. Then a best coapproximation in a cosun $M \subset X$ to an element $x \in \mathfrak{D}_M$ satisfies the inequality

$$g(\|m-y\|) \leqslant g(\|x-y\|) - cg(\|x-m\|) \tag{4.3}$$

for all y in M.

Proof. By (4.1) we have

$$g(\|m-y\|) \leqslant g(\|((x-y)+(m-y))/2\|)$$

for all y in M. Hence by (4.2) we obtain

$$g(\|m-y\|) \leqslant [g(\|x-y\|) + g(\|m-y\|) - cg(\|x-m\|)]/2,$$

which is equivalent to (4.3). ∎

We remark that inequality (4.2) is known for the spaces $X = L_p$ $(1 < p < \infty)$; see Clarkson [3, Theorem 2] and Meir [10, Inequality (2.3)]. In this case we have $g(t) = t^q$ with $q = \max(2, p)$ and

$$c = \begin{cases} 2^{1-p}, & \text{if } 2 \leqslant p < \infty, \\ p(p-1)/4, & \text{if } 1 < p \leqslant 2. \end{cases}$$

The same inequality holds also for the Hardy and Sobolev spaces H^p and $H^{k,p}$ (see [16]). Note that these constants c are larger than the constants c given in (3.9), but smaller than lower estimates for the constants c_p given in (3.3), (3.4). Finally, if X is a uniformly convex space having the modulus of convexity of power type $q \geqslant 2$, then it follows from Lemma 2.1 presented in [13] that inequality (4.2) holds for $c = d/2^{q-1}$ and $g(t) = t^q$, where the positive constant d is defined as in Theorem 3.2.

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